10.2 Second Order Ordinary Differential Equations

1173. Homogeneous Linear Equations with Constant Coefficients

$y'' + py' + qy = 0$.

The characteristic equation is

$\lambda^2 + p\lambda + q = 0$.

If λ_1 and λ_2 are distinct real roots of the characteristic equation, then the general solution is

$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$, where

C_1 and C_2 are integration constants.

If $\lambda_1 = \lambda_2 = -\frac{p}{2}$, then the general solution is

$y = (C_1 + C_2 x)e^{-\frac{p}{2}x}$.

If λ_1 and λ_2 are complex numbers:

$\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, where

$\alpha = -\frac{p}{2}$, $\beta = \frac{\sqrt{4q - p^2}}{2}$,

then the general solution is

$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$.

1174. Inhomogeneous Linear Equations with Constant Coefficients

$y'' + py' + qy = f(x)$.

The general solution is given by

$y = y_p + y_h$, where

y_p is a particular solution of the inhomogeneous equation and y_h is the general solution of the associated homogeneous equation (see the previous topic 1173).

If the right side has the form
$f(x)=e^{\alpha x}\left(P_1(x)\cos\beta x+P_1(x)\sin\beta x\right)$,
then the particular solution y_p is given by
$y_p=x^k e^{\alpha x}\left(R_1(x)\cos\beta x+R_2(x)\sin\beta x\right)$,
where the polynomials $R_1(x)$ and $R_2(x)$ have to be found by using the method of undetermined coefficients.

- If $\alpha+\beta i$ is not a root of the characteristic equation, then the power $k=0$,
- If $\alpha+\beta i$ is a simple root, then $k=1$,
- If $\alpha+\beta i$ is a double root, then $k=2$.

1175. Differential Equations with y Missing
$y''=f(x,y')$.
Set $u=y'$. Then the new equation satisfied by v is
$u'=f(x,u)$,
which is a first order differential equation.

1176. Differential Equations with x Missing
$y''=f(y,y')$.
Set $u=y'$. Since

$$y''=\frac{du}{dx}=\frac{du}{dy}\frac{dy}{dx}=u\frac{du}{dy},$$

we have

$$u\frac{du}{dy}=f(y,u),$$

which is a first order differential equation.

1177. Free Undamped Vibrations
The motion of a Mass on a Spring is described by the equation
$m\ddot{y}+ky=0$,
where
m is the mass of the object,
k is the stiffness of the spring,
y is displacement of the mass from equilibrium.
The general solution is

$y = A\cos(\omega_0 t - \delta)$,
where
A is the amplitude of the displacement,

ω_0 is the fundamental frequency, the period is $T = \frac{2\pi}{\omega_0}$,

δ is phase angle of the displacement.
This is an example of simple harmonic motion.

1178. Free Damped Vibrations
$m\ddot{y} + \gamma\dot{y} + ky = 0$, where
γ is the damping coefficient.
There are 3 cases for the general solution:

Case 1. $\gamma^2 > 4km$ (overdamped)

$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$,

where

$$\lambda_1 = \frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m},\ \lambda_2 = \frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m}.$$

Case 2. $\gamma^2 = 4km$ (critically damped)

$y(t) = (A + Bt)e^{\lambda t}$,

where

$$\lambda = -\frac{\gamma}{2m}.$$

Case 3. $\gamma^2 < 4km$ (underdamped)

$y(t) = e^{-\frac{\gamma}{2m}t} A\cos(\omega t - \delta)$, where

$\omega = \sqrt{4km - \gamma^2}$.

1179. Simple Pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0,$$

where θ is the angular displacement, L is the pendulum length, g is the acceleration of gravity.

The general solution for small angles θ is

$\theta(t) = \theta_{max} \sin\sqrt{\frac{g}{L}}t$, the period is $T = 2\pi\sqrt{\frac{L}{g}}$.

1180. RLC Circuit

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = V'(t) = \omega E_0 \cos(\omega t),$$

where I is the current in an RLC circuit with an ac voltage source $V(t) = E_0 \sin(\omega t)$.

The general solution is

$I(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + A\sin(\omega t - \varphi)$,

where

$$r_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L},$$

$$A = \frac{\omega E_0}{\sqrt{\left(L\omega^2 - \frac{1}{C}\right)^2 + R^2\omega^2}},$$

$$\varphi = \arctan\left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right),$$

C_1, C_2 are constants depending on initial conditions.