## 10.2 Second Order Ordinary Differential Equations

**1173.** Homogeneous Linear Equations with Constant Coefficients y'' + py' + qy = 0.

The characteristic equation is

$$\lambda^2 + p\lambda + q = 0.$$

If  $\lambda_1$  and  $\lambda_2$  are distinct real roots of the characteristic equation, then the general solution is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$
, where

 $C_1$  and  $C_2$  are integration constants.

If  $\lambda_1 = \lambda_2 = -\frac{p}{2}$ , then the general solution is

$$y = (C_1 + C_2 x)e^{-\frac{p}{2}x}$$
.

If  $\lambda_1$  and  $\lambda_2$  are complex numbers:

$$\lambda_1 = \alpha + \beta i$$
,  $\lambda_2 = \alpha - \beta i$ , where

$$\alpha = -\frac{p}{2}$$
,  $\beta = \frac{\sqrt{4q-p^2}}{2}$ ,

then the general solution is

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x).$$

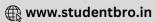
1174. Inhomogeneous Linear Equations with Constant Coefficients

$$y'' + py' + qy = f(x).$$

The general solution is given by

$$y = y_p + y_h$$
, where

 $y_p$  is a particular solution of the inhomogeneous equation and  $y_h$  is the general solution of the associated homogene-



ous equation (see the previous topic 1173).

If the right side has the form

$$f(x) = e^{\alpha x} (P_1(x)\cos\beta x + P_1(x)\sin\beta x),$$

then the particular solution  $y_p$  is given by

$$y_p = x^k e^{\alpha x} (R_1(x) \cos \beta x + R_2(x) \sin \beta x),$$

where the polynomials  $R_1(x)$  and  $R_2(x)$  have to be found by using the method of undetermined coefficients.

- If  $\alpha + \beta i$  is not a root of the characteristic equation, then the power k = 0,
- If  $\alpha + \beta i$  is a simple root, then k = 1,
- If  $\alpha + \beta i$  is a double root, then k = 2.
- 1175. Differential Equations with y Missing

$$y'' = f(x,y').$$

Set u = y'. Then the new equation satisfied by v is

$$\mathbf{u}' = \mathbf{f}(\mathbf{x}, \mathbf{u}),$$

which is a first order differential equation.

**1176.** Differential Equations with x Missing y'' = f(y, y').

Set u = y'. Since

$$y'' = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}$$
,

we have

$$u\frac{du}{dv} = f(y,u),$$

which is a first order differential equation.

1177. Free Undamped Vibrations

The motion of a Mass on a Spring is described by the equation

$$m\ddot{y} + ky = 0$$

where

m is the mass of the object,

k is the stiffness of the spring,

y is displacement of the mass from equilibrium.

The general solution is

$$y = A\cos(\omega_0 t - \delta),$$

where

A is the amplitude of the displacement,

$$\omega_0$$
 is the fundamental frequency, the period is  $T = \frac{2\pi}{\omega_0}$ ,

 $\delta$  is phase angle of the displacement.

This is an example of simple harmonic motion.

## 1178. Free Damped Vibrations

$$m\ddot{y} + \gamma \dot{y} + ky = 0$$
, where

 $\gamma$  is the damping coefficient.

There are 3 cases for the general solution:

Case 
$$1. \gamma^2 > 4 \text{km}$$
 (overdamped)

$$\mathbf{y}(\mathbf{t}) = \mathbf{A}\mathbf{e}^{\lambda_1 \mathbf{t}} + \mathbf{B}\mathbf{e}^{\lambda_2 \mathbf{t}},$$

where

$$\lambda_1=\frac{-\gamma-\sqrt{\gamma^2-4km}}{2m}$$
 ,  $\lambda_2=\frac{-\gamma+\sqrt{\gamma^2-4km}}{2m}$  .

Case 2.  $\gamma^2 = 4$ km (critically damped)

$$y(t) = (A + Bt)e^{\lambda t}$$
,

where

$$\lambda = -\frac{\gamma}{2m}$$
.

Case 3.  $\gamma^2 < 4$ km (underdamped)

$$y(t) = e^{-\frac{\gamma}{2m}t} A \cos(\omega t - \delta)$$
, where  $\omega = \sqrt{4km - \gamma^2}$ .

1179. Simple Pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{I}\theta = 0,$$

where  $\theta$  is the angular displacement, L is the pendulum length, g is the acceleration of gravity.

The general solution for small angles  $\theta$  is

$$\theta(t) = \theta_{max} \sin \sqrt{\frac{g}{L}} t$$
, the period is  $T = 2\pi \sqrt{\frac{L}{g}}$ .

## 1180. RLC Circuit

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = V'(t) = \omega E_0 \cos(\omega t),$$

where I is the current in an RLC circuit with an ac voltage source  $V(t) = E_0 \sin(\omega t)$ .

The general solution is

$$I(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + A \sin(\omega t - \varphi),$$

where

$$\mathbf{r}_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

$$A = \frac{\omega E_0}{\sqrt{\left(L\omega^2 - \frac{1}{C}\right)^2 + R^2\omega^2}},$$

$$\varphi = \arctan\left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right)$$
,

 $C_1$ ,  $C_2$  are constants depending on initial conditions.